Lecture 25

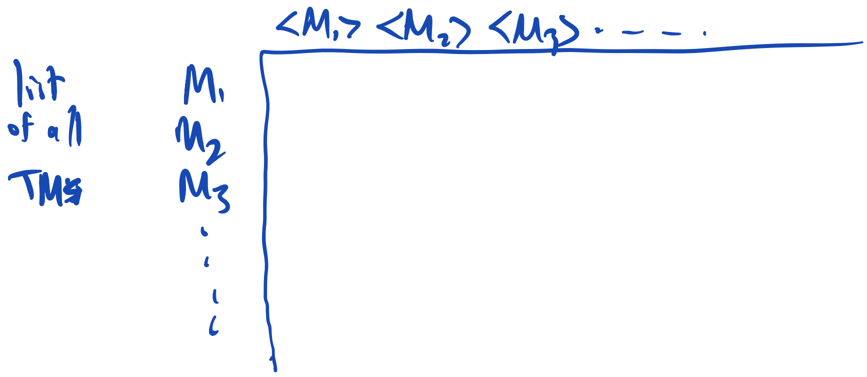
CSE 431
Intro to Theory of Computation

We know:

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP \subseteq NEXP$$

Today we prove these separations:

The method we will prove is based on diagonalization where we designed a new machine that did the opposite of the ith machine M_i on input $\langle M_i \rangle$



We will do something along the same lines for listing all space-bound (or time-bounded TMs)

The construction is similar in the two cases but easier for space.

The general idea is that a bit more space will let TMs to do more, but that only works for "nice" space bounds.

Space Hierarchy Theorem

Def

Defn A function $f: \mathbb{N} \to \mathbb{N}$ is space constructible iff
$f(n) \geq \log_2 n$ and
the map $1^n \mapsto$ binary representation of $f(n)$, $\langle f(n) \rangle$
is computable by an $O(f(n))$-space TM.

The general idea of a listing of all space-bound TMs is that for a space constructible $f(n)$
and any $\langle M \rangle$ we can simulate M on input x
using space $O(f(n))$ s.t.
The simulation does what M does if
- M doesn't use more than $f(|x|)$ storage
- M doesn't run for more than $2^{f(|x|)}$ steps (which implies that M doesn't run forever)

"On input $\langle M \rangle$ and x:
- Use space constructability of f to compute the binary string $\langle f(|x|) \rangle$ on the work tape
  (pretend each symbol of x is a 1)
- Mark off $f(|x|)$ cells on a separate section of the work tape
- Create a counter $2^{f(|x|)}$ using another $f(|x|)$ cells.
- Simulate M on input x keeping track of the # of steps
  - subtract 1 from counter each step
  - stop simulation if A moves off the marked cells & reject
  - stop when counter reaches 0 & reject"

Using this idea we prove

Theorem If $f(n)$ is space constructible Then there is language A decidable using space $O(f(n))$ but not $o(f(n))$.

Proof Define A as the language decided by the following TM for a "diagonal language"

Almost final alg.

On input x:

1. Use space constructibility of f to compute $\langle f(|x|)\rangle$ on the work tape
2. Mark off $f(|x|)$ cells on the work tape
3. If x is not of the form $\langle M\rangle 0 1^n$ then reject
4. Simulate M on input x counting steps
   If more than $2^{f(|x|)}$ steps then stop & accept
   If more than $f(|x|)$ cells used stop & accept
5. If M accepts then reject.
   If M rejects then accept

Here is our fix

Claim A is different from every language decided using space $o(f(n))$

Suppose not. Then $A = L(M_i)$ for some M_i that uses space $g(n) = o(f(n))$

Consider whether A includes $\langle M_i\rangle$

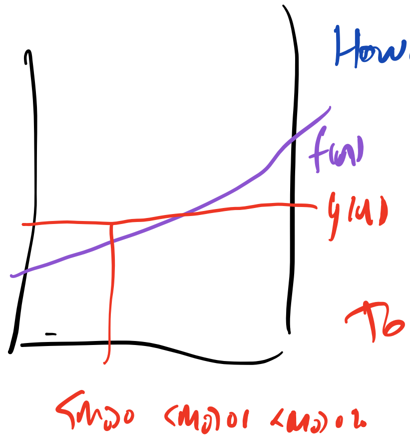
If M_i runs on input $\langle M_i\rangle$

what our fixes do

using $\leq f(|\langle M_i\rangle 0 1^n|)$ cells

and time at most $2^{f(|\langle M_i\rangle 0 1^n|)}$

Then we get a contradiction since we flipped the answer in defining A.



However even though $g(n)$ is $o(f(n))$
$n = \langle M_i \rangle$ might be small enough that $f(n) < g(n)$, in which case there wouldn't be a contradiction.

To get around this we flip an infinite # of values for each M_i, and not just the diagonal.

We use
$x = \langle M_i \rangle 0 1^k$ for all integers k
which still allow us to tell which machine is associated.

Now for any input $x = \langle M_i \rangle 0 1^k$ such that k makes $f(\langle M_i \rangle 0 1^k) \geq g(\langle M_i \rangle 0 1^k)$ is good enough.
and we get a contradiction $\boxtimes$

Cor If $S_1(n)$ is $o(S_2(n))$ then
$SPACE(S_1(n)) \subsetneq SPACE(S_2(n))$

Note : most natural functions are space constructible
n^k, $\log n$, $n \log n$ etc

eg $\log n$: On input 1^n count # of bits onto work tape: gets n in binary which takes $\log n$ bits.
Now count # of bits in that: $\log n$ in binary

Cor $NL \subseteq SPACE(\log^2 n) \subsetneq PSPACE$

Time Hierarchy

Defn $f(n) \geq n \log n$ is <u>time constructible</u>
iff $1^n \to$ binary of $f(n)$ is computable
in time $O(f(n))$

Thm If $t(n)$ is time constructible there
is a language decidable in ~~time~~
$O(t(n))$ but not $o(t(n)/\log t(n))$

possible gap.

Proof idea: Essentially the same as the
one for bounded space except
that on input x
we use ~~time~~ constructability
to compute $t(|x|)$ in binary
and use it as a timer for
the computation.
(count down to 0 subtracting
1 per step)
reject if it exceeds the time

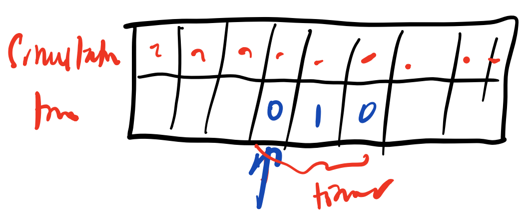
Unlike with space complexity we have
to count # of steps to update
the timer.
The timer takes $\log t(|x|)$ bits
to represent & update

In the course we used multitape TMs for
time. The book used 1-tape TMs
The proof is different in the two cases.

Multitape TM version: We need a fixed # of tapes for the machine defining A but the other TMs M_i might use more tapes:

We use simulation of k-tape TM by 2-tape TM
$t(n)$ steps become $O(t(n)\log t(n))$ steps & it keeps track of a counter

1-tape version: Maintain the counter like a pocket watch that is carried by the TM near the read head:



(think of it as on a separate track of the tape)

shift the timer left or right at each time step.
$O(\log t(n))$ steps per original step.

If $t'(n)$ is $o(t(n)/\log t(n))$ then both of these can be done in $\leq t(n)$ steps.

∎

Can we use these kinds of diagonalization arguments to prove $P \neq NP$?

Idea why "NO": These diagonalization arguments work by "simulation"

We could get separation even if both machines could get access to the answer to some hard problem

"Oracle TMs" M^B gets to ask question: "is y in B" in 1 step
language

Simulation also works for oracle machines.

P^B = languages decidable in poly time using such an oracle TM.

NP^B : same except nondet. oracle TM

If diagonalization showed $P \neq NP$ it would also show $P^B \neq NP^B$ for every B

But: $\exists B .\ P^B = NP^B$

Example $B = TQBF$

$$P^{TQBF} = PSPACE = NPSPACE = NP^{TQBF}$$

TQBF is PSPACE-complete

Savitch

Same idea

Fact: Also $\exists A$. one can prove $P^A \neq NP^A$
(complicated see 9.2 in text)

Q: why wouldn't SAT work?

NP^{SAT} can decide formulas of the form:

$$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m \varphi(x_1 \ldots x_n, y_1 \ldots y_m)$$

Here's how: NP machine guesses $x_1 \ldots x_n = b_1 \ldots b_n$
then calls SAT oracle on

$$\neg \varphi(b_1, \ldots b_n, y_1 \ldots y_m)$$

(if this is not SAT then

$$\forall y_1 \ldots \forall y_m \varphi(b_1 \ldots b_n, y_1 \ldots y_m)$$

is true

so the whole formula would be true)

$$\therefore \Sigma_2^P \subseteq NP^{SAT}$$

However $P^{NP} \subseteq \Sigma_2^P \cap \Pi_2^P$

we don't know if $\Sigma_2^P = \Pi_2^P$ (researchers guess not)

So it is possible that $\Sigma_2^P \cap \Pi_2^P \neq \Sigma_2^P$

in which case $P^{SAT} \neq NP^{SAT}$.